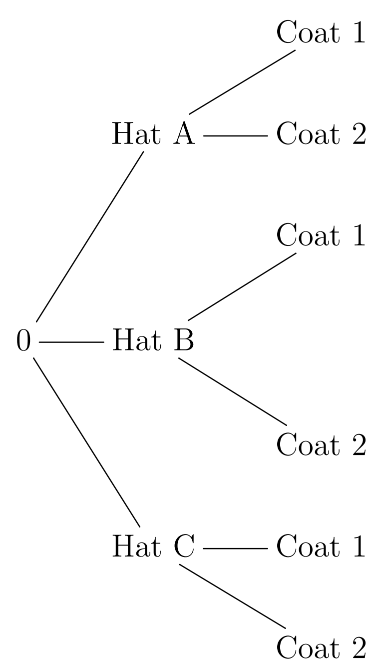
Section 5.4 : The Multiplication Principle

Two step multiplication principle: Assume that a task can be broken up into two consecutive steps. If step 1 can be performed in m ways and for each of these, step 2 can be performed in n ways, then the task itself can be performed in $m \times n$ ways.

Example 1 If you have 3 hats, hats A, B and C, and 2 coats, Coats 1 and 2, in your closet. Assuming that you feel comfortable with wearing any hat with any coat, How many different choices of hat/coat combinations do you have? List all combinations.

We can get some insight into why the formula holds by representing all options on a tree diagram. We can break the decision making process into two steps here: Step 1: Choose a hat, Step 2: choose a coat. From the starting point 0, we can represent the three choices for step 1 by three branches whose endpoints are labelled by the choice names. From each of these endpoints we draw branches representing the options for step two with endpoints labelled appropriately. The result for the above example is shown below:



Another solution

Each path on the tree diagram corresponds to a choice of hat and coat. Each of the three branches in step 1 is followed by two branches in step 2, giving us 3×2 distinct paths.

If we had m hats and n coats, we would get $m \times n$ paths on our diagram. Of course if the numbers m and n are large, it may be difficult to draw.

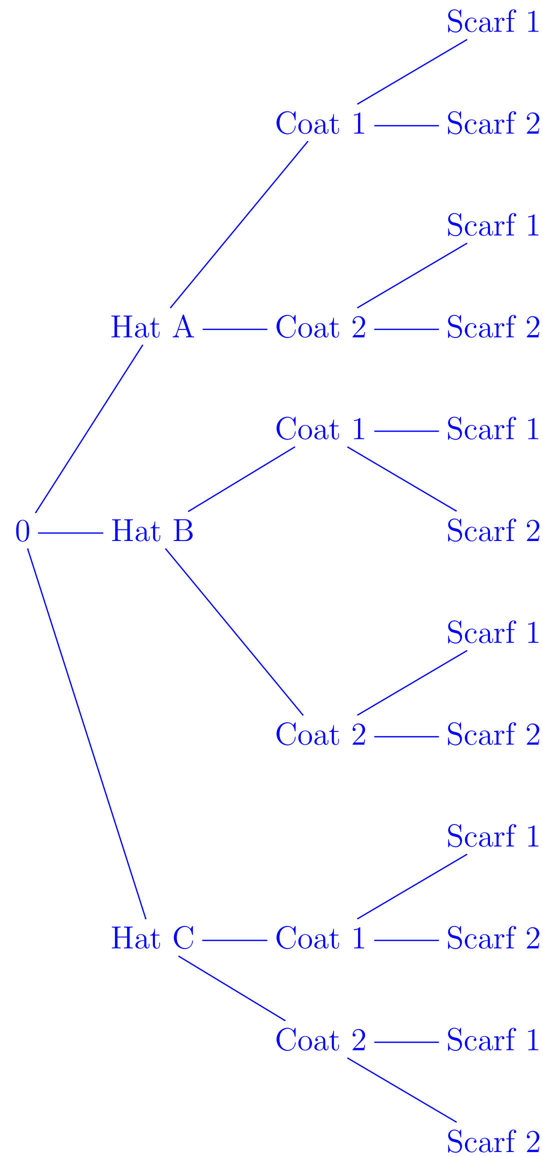
Example 2 The South Shore line runs from South Bend Airport to Randolph St. Station in Chicago. There are 20 stations at which it stops along the line. How many one way tickets could be printed, showing a point of departure and a destination?

You can start at any of twenty stations. Once this is picked, you can pick any of nineteen destinations. The answer is $20 \cdot 19 = 380$.

Example 3 You want to design a 30 minute workout. For the first 15 minutes, you will choose an aerobic exercise from running, kickboxing, skipping or circuit training. For the second 15 minutes, you will work on strength and/or balance choosing from weight training, TRX, Bosu, resistance bands or your core routine. How many such workouts are possible.

There are 4 things you can do for your first 15 minutes. There are 5 things you can do for the second 15 minutes. The answer is $4 \cdot 5 = 20$.

Example 4 If your closet contains 3 hats, 2 coats and 2 scarves. Assuming you are comfortable with wearing any combination of hat, coat and scarf, (and you need a hat, coat and scarf today), how many different outfits could you select from your closet? (Break the decision making process into steps and draw a tree diagram representing the possible choices.)


0
Hat A
Hat B
Hat C
Coat 1
Coat 2
Coat 1
Coat 2
Coat 1
Coat 2
Scarf 1
Scarf 2
Scarf 1
Scarf 2
Scarf 1
Scarf 2
Scarf 1
Scarf 2
Scarf 1
Scarf 2
Scarf 1
Scarf 2

The General Multiplication Principle

If a task can be broken down into R consecutive steps, Step 1, Step 2, ..., Step R, and if
I can perform step 1 in m_1 ways,
and for each of these I can perform step 2 in m_2 ways,
and for each of these I can perform step 3 in m_3 ways,
and so forth

Then the task can be completed in

$$m_1 \times m_2 \times . \cdots \times m_R$$

ways.

Example 5 How many License plates, consisting of 2 letters followed by 4 digits are possible? Would this be enough for all the cars in Indiana?
(Note that it is not a good idea to try to solve this with a tree diagram).

There are 26 letters and 10 digits so the answer is

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$$

The current population of Indiana seems to be just short of $6,500,000$.

Example 6 A group of 5 boys and 3 girls is to be photographed.

(a) How many ways can they be arranged in one row?

There are 8 people so there are

$$8 \cdot 7 \cdot \cdots \cdot 2 \cdot 1 = 8! = 40,320$$

possible ways to do this. The fact that some of them are boys and others girls is irrelevant.

(b) How many ways can they be arranged with the girls in front and the boys in the back row?

There are 3 girls so there are $3 \cdot 2 \cdot 1 = 3!$ ways to arrange the first row. There are 5 boys so there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways to arrange the second row. The two rows can be arranged independently so the answer is $3! \cdot 5! = 6 \cdot 120 = 720$ possibilities.

Example 7 How many different 4 letter words (including nonsense words) can you make from the letters of the word

MATHEMATICS

if (a) letters cannot be repeated (MMMM is not considered a word but MTCS is).

'MATHEMATICS' has 8 distinct letters $\{$M, A, T, H, E, I, C, S$\}$.
Hence the answer is $8 \cdot 7 \cdot 6 \cdot 5 = 1,680$

(b) letters can be repeated (MMMM is considered a word).

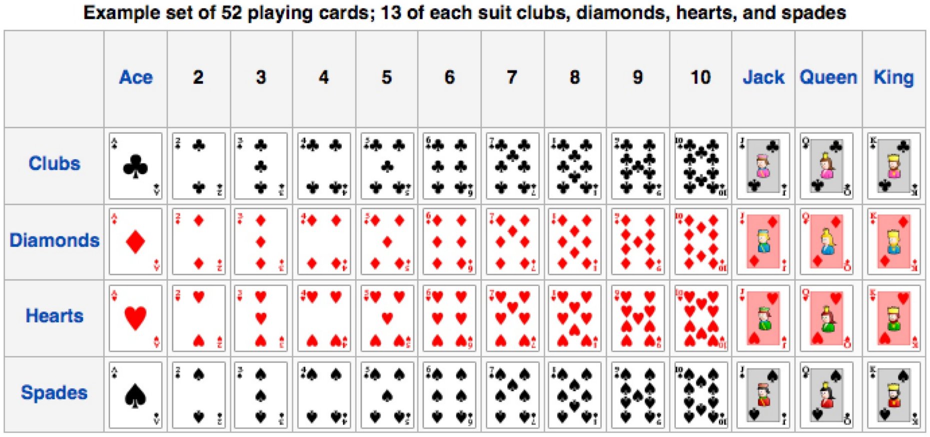
There are still only 8 distinct letters so the answer is $8 \cdot 8 \cdot 8 \cdot 8 = 8^4 = 4,096$.

(c) Letters cannot be repeated and the word must start with a vowel.

The 8 distinct letters $\{$M, A, T, H, E, I, C, S$\}$ have 3 vowels $\{$A, E, I$\}$. You can select a vowel in any of 3 ways. Once you have done this you have 7 choices for the second letter; 6 choices for the third letter; and 5 choices for the fourth letter. Hence the answer is $3 \cdot 7 \cdot 6 \cdot 5 = 630$.

A standard deck of 52 cards can be classified according to suits or denominations as shown in the picture from Wikipedia below. We have 4 suits, Hearts Diamonds, Clubs and Spades and 13 denominations, Aces, Kings, Queens, . . . , twos.

Example set of 52 playing cards; 13 of each suit clubs, diamonds, hearts, and spades



Example 8 Katy and Peter are playing a card game. The dealer will give each one card and the player will keep the card when it is dealt to them.

(a) How many different outcomes can result.

(b) In how many of the possible outcomes do both players have Hearts?

(a) $52 \cdot 51$

(b) $13 \cdot 12$

Combining Counting Principles Recall that the inclusion-exclusion principle says that if A and B are sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

If the sets A and B are **disjoint** then this principle reduces to $n(A \cup B) = n(A) + n(B)$. Thus in counting disjoint sets, we can just count the number of elements in each and add. This principle extends easily to $R > 2$ disjoint sets:

If $A_1, A_2, \ldots A_R$ are disjoint sets, then $n(A_1 \cup A_2 \cup \cdots \cup A_n) = n(A_1) + n(A_2) + \cdots + n(A_R)$.

Example 9 Katy and Peter are playing a card game. The dealer will give each one card and the player will keep the card when it is dealt to them. In how many of the possible outcomes do both players have cards from the same suit?

There are four distinct possibilities. The possibilities are 2 clubs, 2 diamonds, 2 hearts or 2 spades and these are distinct. In each of these the first card has 13 possibilities while the second has 12. Hence the answer is $(13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12)$.

A second approach is that there are 52 ways to pick the first card and then there are 12 ways to pick the second. Hence the answer is $52 \cdot 12$.

We can **rephrase the above additive principle** in terms of carrying out a task:
Suppose a task can be carried out in R different ways using one of R activities

$$A_1 = \text{Activity } 1,\ A_2, \ldots, A_R.$$

Suppose also that no two of these activities can be performed simultaneously and that activity i, A_i can be performed in $n(A_i)$ ways, then the task can be carried out in $n(A_1) + n(A_2) + \cdots + n(A_R)$ ways .
[**Note** that $A_1, A_2, \ldots, A_R$ are not consecutive steps in the process of completing this task, you must choose only one of them to perform the task.]

This is really a simple everyday principle in disguise, and it will make more sense when you think through this problem:

Example 10 Suppose you are going to buy a single carton of milk today. You can either buy it on campus when you are at school, or at the mall when you go to get a gift for a friend or in the neighborhood near your apartment on your way home. There are 5 different shops on campus to buy from, 2 at the mall and 3 in your neighborhood. In how many different shops can you buy the milk?

There are three distinct outcomes. You buy the milk on campus with 5 choices, or you buy the milk at the mall with 2 choices or you buy the milk in your neighborhood with 3 choices, so the answer is $5 + 2 + 3$.

If you answered $5 \cdot 2 \cdot 3$ you answered the question of how many ways could you buy one carton of milk on campus, one carton at the mall and one carton near home. In particular you end up with three cartons.

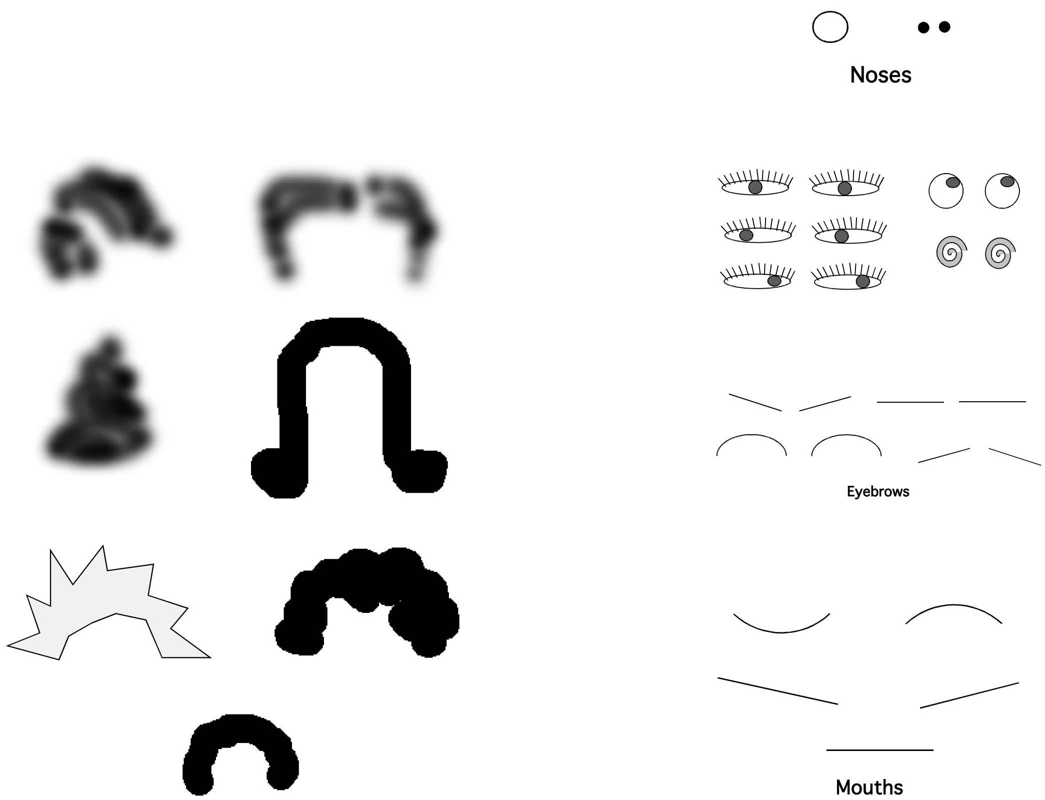
Example 11 Suppose you wish to photograph 5 schoolchildren on a soccer team. You want to line the children up in a row and Sid insists on standing at the end of the row(either end will do). If this is the only restriction, in how many ways can you line the children up for the photograph? (You can think through this as the number of ways to carry out the task or the number of photographs in a set).

There are two distinct possibilities, Sid is on the left or Sid is on the right. There are 4! ways to arrange the other children. Hence the answer is $4! + 4!$.

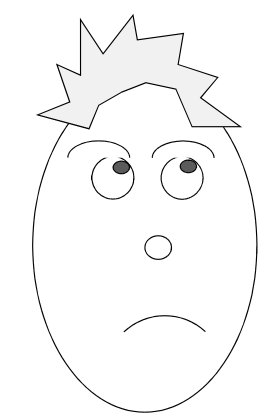
Extras, Multiplication Principle

How many faces can you make?

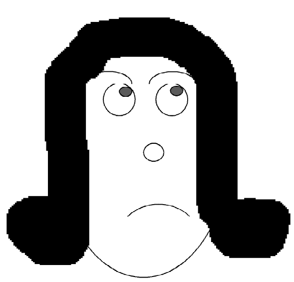
Below you are given 5 pairs of eyes, 4 sets of eyebrows, 2 noses, 5 mouths and 7 hairstyles to choose from. How many possible faces can you make using combinations of the features given if each face you make has a pair of eyes, a pair of eyebrows, a nose, a mouth, and one of the given hairstyles?



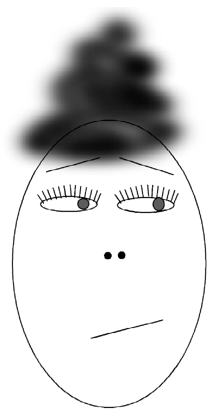
Here is an example of 3 faces, draw three different faces with the features given!



I don't want a Lisa Simpson Hairdo!



If you say "Multiplication Principle" one more time.....



How many roads must a face walk down.....

How many insults can you make?

If you follow the directions on the following Shakespeare Insult Kit, how many different insults can you make?

Shakespeare Insult Kit

To create a Shakespearean insult...

Combine one word from each of the three columns below, prefaced with "Thou":

Column 1	Column 2	Column 3
artless	base-court	apple-john
bawdy	bat-fowling	baggage
beslubbering	beef-witted	barnacle
bootless	beetle-headed	bladder
churlish	boil-brained	boar-pig
cockered	clapper-clawed	bugbear
clouted	clay-brained	bum-bailey
craven	common-kissing	canker-blossom
currish	crook-pated	clack-dish
dankish	dismal-dreaming	clotpole
dissembling	dizzy-eyed	coxcomb
droning	doghearted	codpiece
errant	dread-bolted	death-token
fawning	earth-vexing	dewberry
fobbing	elf-skinned	flap-dragon
froward	fat-kidneyed	flax-wench
frothy	fen-sucked	flirt-gill
gleeking	flap-mouthed	foot-licker
goatish	fly-bitten	fustilarian
gorbellied	folly-fallen	giglet
impertinent	fool-born	gudgeon
infectious	full-gorged	haggard
jarring	guts-griping	harpy
loggerheaded	half-faced	hedge-pig
lumpish	hasty-witted	horn-beast
mammering	hedge-born	hugger-mugger
mangled	hell-hated	joithead
mewling	idle-headed	lewdster
paunchy	ill-breeding	lout
pribbling	ill-nurtured	maggot-pie
puking	knotty-pated	malt-worm
puny	milk-livered	mammet
qualling	motley-minded	measle
rank	onion-eyed	minnow
reeky	plume-plucked	miscreant
roguish	pottle-deep	moldwarp
ruttish	pox-marked	mumble-news
saucy	reeling-ripe	nut-hook
spleeny	rough-hewn	pigeon-egg
spongy	rude-growing	pignut
surly	rump-fed	puttock
tottering	shard-borne	pumpion
unmuzzled	sheep-biting	ratsbane
vain	spur-galled	scut
venomed	swag-bellied	skainsmate
villainous	tardy-gaited	strumpet
warped	tickle-brained	varlot
wayward	toad-spotted	vassal
weedy	unchin-snouted	whey-face
yeasty	weather-bitten	wagtail

Old Exam Questions For Review

1 Five square tiles of the same size but of different colors (all 5 colors are different) are arranged side by side in a horizontal line. How many different patterns are possible?
(a) 2^5 (b) 5 (c) 5^2 (d) 120 (e) 100

2 Piraullis pizza joint offers a mix and match pizza on its menu. There are 4 different meats to choose from, 5 different vegetables, 4 different types of cheese, and 2 different types of crust. How many different types of Pizza can be made by choosing 1 type of meat, 1 vegetable, 1 cheese and 1 crust?
(a) 80 (b) 4 (c) 20 (d) 160 (e) 49